Solution Sheet 4

In this problem sheet, unless otherwise stated, for a Gaussian measure μ on $\mathbb{R}^n$ we fix $m \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ such that for all $\lambda \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle}\mu(dx) = e^{i\langle \lambda, m\rangle - \frac{1}{2}\langle K\lambda,\lambda\rangle}. \tag{1}$$

We also define the Fourier Transform of a Borel Measure ν on $\mathbb{R}^n$ by

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle}\nu(dx) \tag{2}$$

and for ν on a Banach Space E, $\hat{\nu} : E^* \to \mathbb{C}$, by

$$\hat{\nu}(l) = \int_E e^{il(x)}\nu(dx).$$

Exercise 4.1.

For $y \in \mathbb{R}^n$, prove that δ_y is a Gaussian measure. If $y \in E$ for a Banach Space E, is δ_y a Gaussian measure?

Proof. By definition for $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{i\langle \lambda, x\rangle}\delta_y(dx) = e^{i\langle \lambda, y\rangle}$$

hence δ_y satisfies (1) for $m = y$, $K = 0$. The second part is true: we are required to prove that for every $l \in E^*$, the push-forward measure $l_*\delta_y$ is a Gaussian measure on $\mathbb{R}$. Indeed,

$$\int_{\mathbb{R}} e^{i\langle \lambda, x\rangle} l_*\delta_y(dx) = \int_{\mathbb{R}} e^{i\langle \lambda, l(x)\rangle}\delta_y(dx) = e^{i\langle \lambda, l(y)\rangle}$$

hence $l_*\delta_y$ satisfies (1) for $m = l(y)$, $K = 0$. □

Exercise 4.2.

Let μ, ν be probability measures on a separable Banach Space E.

1. Show that if $l_*\mu = l_*\nu$ for all $l \in E^*$, then $\mu = \nu$.

   Hint: As a consequence of the Hahn-Banach Separation Theorem, every closed ball $B \subset E$ admits the representation $B = \bigcap_{i\in I} A_i$ for some countable indexing set I, and for sets A_i of the form $A_i = \{x \in E : l_i(x) \leq c\}$.

2. Prove that if $\hat{\mu}(l) = \hat{\nu}(l)$ for all $l \in E^*$, then $\mu = \nu$.

Proof.

1. We assume that $l_*\mu = l_*\nu$ for all $l \in E^*$, then considering Borel sets $(-\infty, c]$ in $\mathbb{R}$, we have that $\mu\left(l^{-1}(-\infty, c])\right) = \nu\left(l^{-1}(-\infty, c])\right)$ or in other words, $\mu(\{x \in E : l(x) \leq c\}) = \nu(\{x \in E : l(x) \leq c\})$. Therefore all sets of the form A_i as in the hint satisfy $\mu(A_i) = \nu(A_i)$, so all closed balls $B \subset E$ are such that $\mu(B) = \nu(B)$. Closed balls generate the Borel $\sigma-$algebra, from which standard $\pi - \lambda$ theory allows us to conclude that the measures are equal.

2. By assumption,
$$\int_E e^{il(x)}\mu(dx) = \int_E e^{il(x)}\nu(dx)$$
for all $l \in E^*$. Note that for any $\xi \in \mathbb{R}$ we have that $\xi l \in E^*$, defined by $(\xi l)(x) = \xi l(x)$, hence
$$\int_E e^{i\xi l(x)}\mu(dx) = \int_E e^{i\xi l(x)}\nu(dx)$$
for all $l \in E^*$ and $\xi \in \mathbb{R}$. Taking the push-forward measure in each integral,
$$\widehat{l_*\mu}(\xi) = \int_{\mathbb{R}} e^{i\xi x} l_*\mu(dx) = \int_E e^{i\xi x} l_*\nu(dx) = \widehat{l_*\nu}(\xi).$$
Now we use that the Fourier transform on $\mathbb{R}$ determines the measure (Proposition 2.4.2), hence $l_*\mu = l_*\nu$ for all $l \in E^*$, so by the previous part $\mu = \nu$.

□

Exercise 4.3.

1. Let $\{X_1, \dots, X_N\}$ be independent random variables such that each X_j is Gaussian on $\mathbb{R}^n$, and $a_j \in \mathbb{R}$. Show that $\sum_{j=1}^N a_j X_j$ is Gaussian on $\mathbb{R}^n$.
2. Let $\{X_1, \dots, X_N\}$ be independent random variables such that each X_j is Gaussian on $\mathbb{R}$, and $a_j \in E$ for some Banach Space E. Show that $\sum_{j=1}^N a_j X_j$ is Gaussian on E.

Proof.

1. Each X_j is Gaussian means, by definition, for μ_j the law of X_j,
$$\mathbb{E}\left(e^{i\langle\lambda, X_j\rangle}\right) = \int_{\mathbb{R}^n} e^{i\langle\lambda, x\rangle}\mu_j(dx) = e^{i\langle\lambda, m_j\rangle - \frac{1}{2}\langle K_j\lambda,\lambda\rangle} \tag{3}$$
for some m_j, K_j. For $\sum_{j=1}^N a_j X_j$ we use $e^{i\langle\lambda, \sum_{j=1}^N a_j X_j\rangle} = e^{i\sum_{j=1}^N a_j\langle\lambda, X_j\rangle} = \Pi_{j=1}^N e^{i\langle a_j\lambda, X_j\rangle}$ such that, at first by independence and then by (3),
$$\begin{aligned}\mathbb{E}\left(e^{i\langle\lambda, \sum_{j=1}^N a_j X_j\rangle}\right) &= \Pi_{j=1}^N \mathbb{E}\left(e^{i\langle a_j\lambda, X_j\rangle}\right)\\ &= \Pi_{j=1}^N e^{i\langle a_j\lambda, m_j\rangle - \frac{1}{2}\langle K_j a_j\lambda, a_j\lambda\rangle}\\ &= \Pi_{j=1}^N e^{i\langle \lambda, m_j a_j\rangle - \frac{1}{2}\langle a_j^2 K_j\lambda, \lambda\rangle}\\ &= e^{i\langle\lambda, \sum_{j=1}^N m_j a_j\rangle - \frac{1}{2}\langle \sum_{j=1}^N a_j^2 K_j\lambda,\lambda\rangle}\end{aligned}$$
as required.
2. Defining $X = \sum_{j=1}^N a_j X_j$ and μ the law of X on E, for any given $l \in E^*$ we consider
$$\int_{\mathbb{R}} e^{i\lambda x} l_*\mu(dx) = \int_{\mathbb{R}} e^{i\lambda l(x)}\mu(dx) = \mathbb{E}\left(e^{i\lambda l(X)}\right).$$

Similarly to the previous part we break the last term up,

$$\begin{aligned}\mathbb{E}\left(e^{i\lambda l(X)}\right) &= \mathbb{E}\left(e^{i\lambda l\left(\sum_{j=1}^{N} a_j X_j\right)}\right)\\ &= \mathbb{E}\left(e^{\sum_{j=1}^{N} i\lambda l(a_j) X_j}\right)\\ &= \Pi_{j=1}^{N}\mathbb{E}\left(e^{i\lambda l(a_j) X_j}\right)\\ &= \Pi_{j=1}^{N} e^{i\lambda l(a_j) m_j - \frac{1}{2}(l(a_j))^2 K_j \lambda^2}\\ &= e^{i\lambda \sum_{j=1}^{N} l(a_j) m_j - \frac{1}{2}\sum_{j=1}^{N}(l(a_j))^2 K_j \lambda^2}\end{aligned}$$

demonstrating that $l_*\mu$ is Gaussian for arbitrary $l \in E^*$, concluding the proof.

□

Exercise 4.4.

Consider a sequence of real numbers (ε_n) convergent to zero as $n \to \infty$, and corresponding Gaussian measures μ_n on $\mathbb{R}$ with mean m and variance ε_n^2. Prove that (μ_n) converges weakly to δ_m as $n \to \infty$.

Proof. For any bounded and continuous $f : \mathbb{R} \to \mathbb{R}$, we are required to prove that

$$\int_{\mathbb{R}} f d\mu_n \longrightarrow \int_{\mathbb{R}} f d\delta_m = f(m).$$

Using the explicit form of the density for μ_n,

$$\int_{\mathbb{R}} f d\mu_n = \frac{1}{\varepsilon_n\sqrt{2\pi}}\int_{\mathbb{R}} f(x) e^{-\frac{(x-m)^2}{2\varepsilon_n^2}} dx$$

and with the substitution $y = \frac{x-m}{\varepsilon_n}$, this is further reduced to

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\varepsilon_n y + m) e^{\frac{-y^2}{2}} dy.$$

The integrand is pointwise convergent to $f(m)e^{\frac{-y^2}{2}}$ due to continuity of f, and as f is bounded we can freely apply the Dominated Convergence Theorem (with dominating function $\sup_{z\in\mathbb{R}}|f(z)|e^{\frac{-y^2}{2}}$) to deduce that this integral converges to

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(m) e^{\frac{-y^2}{2}} dy = f(m)$$

as required.

□

Exercise 4.5.

Let (e_j) be a basis of a separable Hilbert Space H, (Y_j) a collection of independent standard real-valued Gaussian random variables (mean zero and variance one), and $X_n = \sum_{j=1}^{n} e_j Y_j$. We use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the inner product and norm on H, respectively.

1. Show that the sequence of real-valued functions $(\widehat{\mu_n}(\cdot))$ on H converges pointwise to $e^{-\frac{1}{2}\|\cdot\|^2}$.
2. Prove that Lévy's Continuity Theorem, Theorem 2.4.6, does not hold if one replaces $\mathbb{R}^d$ by a general separable Hilbert Space H.

Proof. We answer the questions in turn:

1. Recall the definition of the Fourier transform for $z \in H$,
$$\widehat{\mu_n}(z) = \int_{\mathbb{R}} e^{i\langle z,x\rangle} \mu_n(dx) = \mathbb{E}\left(e^{i\langle z,X_n\rangle}\right).$$
Now exactly as in Exercise 4.3 part 2, with $\lambda = 1$, $l(x) = \langle z, x\rangle$, $m_j = 0$ and $K_j = 1$, this quantity is simply
$$e^{-\frac{1}{2}\sum_{j=1}^n \langle z,e_j\rangle^2}.$$
Taking the limit as $n \to \infty$ gives the result.

2. We have shown that the sequence of Fourier transforms $(\widehat{\mu}_n)$ converges pointwise to a function which is continuous at zero. To prove that the theorem does not hold in this context, we show that (μ_n) cannot converge weakly. We claim that it is sufficient to show that the collection (μ_n) is not tight: indeed if (μ_n) is weakly convergent then it is certainly relatively compact, hence tight by Prohorov's Theorem (Theorem 2.3.8). We demonstrate that (μ_n) cannot be tight by showing that for every $K \subset H$ compact, $\limsup_{n\to\infty} \mu_n(K) = 0$. Recall that every compact K is contained in some closed ball around the origin, B_R. Then
$$\mu_n(B_R) = \mathbb{P}(\|X_n\| \le R) = \mathbb{P}\left(\|X_n\|^2 \le R^2\right) = \mathbb{P}\left(\sum_{j=1}^n Y_j^2 \le R^2\right).$$
Note that $\sum_{j=1}^n Y_j^2$ is a chi-squared distribution with n degrees of freedom, hence this expression is just the cumulative distribution function of the chi-squared distribution, which is known to approach zero as n goes to infinity.

□

Exercise 4.6

Let T be a positive symmetric linear operator on a separable Hilbert Space. Prove that its trace as in Definition 2.6.6 is independent of the choice of basis.

Proof. Consider two different bases for H, (e_n) and (a_k). Recall that as T is positive and symmetric, there exists a positive symmetric linear operator $\sqrt{T}$ such that $\left(\sqrt{T}\right)^2 = T$. By direct calculation,
$$\sum_{n=1}^\infty \langle Te_n, e_n\rangle = \sum_{n=1}^\infty \langle \sqrt{T}\sqrt{T}e_n, e_n\rangle = \sum_{n=1}^\infty \langle \sqrt{T}e_n, \sqrt{T}e_n\rangle = \sum_{n=1}^\infty \|\sqrt{T}e_n\|^2 = \sum_{n=1}^\infty\sum_{k=1}^\infty \langle \sqrt{T}e_n, a_k\rangle^2.$$
Note that we can change the order of summation by Fubini-Tonelli as the summand is non-negative. Thus we further reduce this to
$$\sum_{k=1}^\infty\sum_{n=1}^\infty \langle \sqrt{T}e_n, a_k\rangle^2 = \sum_{k=1}^\infty\sum_{n=1}^\infty \langle e_n, \sqrt{T}a_k\rangle^2 = \sum_{k=1}^\infty \|\sqrt{T}a_k\|^2 = \sum_{k=1}^\infty \langle Ta_k, a_k\rangle$$
as required.

□